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Constraints of the 2+1 dimensional integrable soliton systems

Yi Cheng and Yi-shen Li

Department of Mathematics, University of Science and Technology of China, Hefei 230026, Anhui, People's Republic of China

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Abstract. We show that the linear systems associated with some integrable hierarchies of the soliton equations in 2+1 dimensions can be constrained to integrable hierarchies in 1+1 dimensions such that submanifolds solutions of the given systems in 2+1 can be obtained by solving the resulting integrable systems in 1+1 dimensions. The constraints of the KP hierarchy to the AKNS and Burgers hierarchies respectively are shown in detail and the results of these for the modified KP and 2+1 dimensional analogue of the Caudrey-Dodd-Gibbon-Kotera-Sawata equations to several integrable systems in 1+1 are given.

1. Introduction

It has been shown that many finite dimensional integrable systems naturally arise as the restriction of integrable hierarchy of the soliton equations in one spatial and one temporal (i.e. 1+1) dimensions to a finite dimensional manifold invariant with respect to all equations in the hierarchy (see e.g. [1-4]). A well known example is the constraint of the Korteweg-de Vries (KdV) hiearchy to the pure multisolitons submanifold [1, 2]. In this case, the constraint is realized by the identification of potential of the KdV hiearchy to the squared eigenfunction and the resulting finite dimensional systems for th eigenfunction as the dynamical variable can be proved to be completely integrable in the sense of the Liouville theorem [3, 4].

Recently, the above study has been generalized to the soliton equations in two spatial and one temporal (i.e. 2+1) dimensions [5, 6]. For instance, by identifying the potential u (i.e. the simplest conserved covariant) of the following Kadomtsev-Petviashvili (KP) equation

$$u_{t_3} = a_3(u) = \tfrac{3}{4}\partial_x^{-1}u_{yy} + \tfrac{1}{4}u_{xxx} + \tfrac{3}{2}uu_x \tag{1.1}$$

to its squared eigenfunction $\varphi\varphi^*$ (i.e. the conserved covariant generator), the associated linear systems for φ and their adjoints for φ^* are constrained to the 1+1 dimensional integrable system consisting of the generalized nonlinear Schrödinger (NS) and the generalized modified KdV (MKdV) equations [5, 6]. These equations are the first two non-trivial ones in the AKNS hiearchy. As a consequence, one is able to obtain solutions of the KP equation by solving the resulting integrable system in 1+1 dimensions.

The main purpose of the present paper is to investigate the constraints for the linear systems associated with all the integrable hiearachy of equations in 2+1 dimensions. First of all, we show that the constaint $u = -2\varphi\varphi^*$ of [5, 6] on the linear systems and their adjoints of the KP hierarchy leads to the whole of the AKNS hierarchy for dynamical variables φ and φ^*. This constraint is a natural generalization of that for the KdV hierarchy; however, it is by no means unique in the sense that a constraint leads the integrable system in 2+1 to the integrable systems in 1+1 dimensions. In section 4, we show that the other constraint $u=2\varphi_x$ on the potential and the eigenfunction leads the linear systems of the KP hierarchy to the 1+1 dimensional Burgers hierarchy which is linearizable by the Cole–Hopf transformation. An important significance of these two constraints is that the submanifolds solutions of the KP hierarchy can be constructed by solving the resulting 1+1 dimensional integrable systems. Here we are not going to display these solutions of the KP hierarchy obtained in this way—the interested reader can find them in our previous papers [5, 7]. In section 5, we show the similar constraint of other 2+1 dimensional integrable systems including the modified Kadomtsev–Petviashvili (MKP) and the 2+1 dimensional analogue of the Caudrey–Dodd–Gibbon–Kotera–Sawada (CDGKS) equations, and give concluding remarks in the last section.

Since we will describe the constraints of the KP systems in detail in section 2 we give a brief survey of the construction of the KP hierarchy and the associated linear systems in terms of the mastersymmetry approach.

2. The KP hierarchy

The KP hierarchy constructed in terms of the mastersymmetry approach is the following class of commuting flows (see e.g. [8–10])

$$u_{t_n}=a_n(u) \tag{2.1}$$

where

$$\begin{aligned}&a_1(u)=u_x \qquad a_2(u)=u_y \qquad a_3(u) \text{ is in } (1.1)\\&a_4(u)=\tfrac{1}{2}\partial_x^{-2}u_{yyy}+u_x\partial_x^{-1}u_y+\tfrac{1}{2}u_{xxy}+2uu_y\end{aligned} \tag{2.2}$$

and the other higher-order ones are derived by

$$a_{n+1}(u)=\frac{1}{n}(a_n'[b]-b'[a_n]) \tag{2.3}$$

with

$$b(u)=2ya_3(u)+xa_2(u)+2\partial_x^{-1}u_y \tag{2.4}$$

being the first non-trivial mastersymmetry [8–10]. $a_n'[b]$, etc. in (2.3) denotes the Gateaux derivative of $a_n(u)$

$$a_n'[b]=\lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon}a_n(u+\varepsilon b) \tag{2.5}$$

in the direction $b(u)$, etc.

For $n=3$, (2.1) is the KP equation (1.1). All the equations in (2.1) are the compatibility conditions of the following linear systems [11, 12]:

$$L\varphi=0 \qquad L=\partial_y+\partial_x^2+u \tag{2.6}$$

$$\varphi_{t_n}=A_n\varphi \tag{2.7}$$

where the Lax operators are the polynomials in ∂_x such that

$$a_n(u) = [A_n, L] \tag{2.8}$$

identically. These Lax operators are derived as follows:

$$\begin{aligned} &A_1(u) = \partial_x \qquad A_2(u) = -(\partial_x^2 + u) \\ &A_3(u) = \partial_x^3 + \tfrac{3}{2}u\partial_x + \tfrac{3}{4}(u_x - \partial_x^{-1}u_y) \end{aligned} \tag{2.9}$$

and the others are given by

$$A_{n+1} = \frac{1}{n}(A_n'[b] - B'[a_n] + [A_n, B]) \tag{2.10}$$

where

$$B(u) = 2yA_3 + xA_2 - \partial_x - \tfrac{1}{2}\partial_x^{-1}u \tag{2.11}$$

satisfying

$$b(u) = [B, L] \tag{2.12}$$

identically. The operator $B(u)$ in (2.11) is also a polynomial in ∂_x but with the leading term's coefficient being proportional to y. It generates all A_n in the same way as the mastersymmetry $b(u)$ of (2.4) does in (2.3).

For an operator $A(u) = \Sigma\, a_j(u)\partial_x^i\partial_y^j$, if we define its adjoint formally by $A^*(u) = \Sigma\,(-1)^{i+j}\partial_y^i \cdot a_j(u)$, then the adjoints of the linear systems read

$$L^*\varphi^* = 0 \qquad L^* = \partial_y - \partial_x^2 - u \tag{2.13}$$

$$(\varphi^*)_{t_n} = -A_n^*\varphi^*. \tag{2.14}$$

The operators A_n^* also satisfy

$$a_n(u) = [L^*, A_n^*]. \tag{2.15}$$

They can be generated independently by

$$A_{n+1}^* = \frac{1}{n}((A_n^*)'[b] - (B^*)'[a_n] + [B^*, A_n^*]) \tag{2.16}$$

with

$$B^* = 2yA_3^* + xA_2^* - \partial_x - \tfrac{1}{2}(\partial_x^{-1}u). \tag{2.17}$$

The quantity $\varphi\varphi^*$ is the well known squared eigenfunction of the KP system. It also solves the equation [13]

$$\tau_{t_n} = -(a_n')^*[\tau] \tag{2.18}$$

for $\tau(u)$, thus $\varphi\varphi^*$ is the conserved covariant generator of the KP hierarchy (2.1).

3. Constraint of the KP hierarchy to the AKNS hierarchy

Let us identify the simplest conserved covariant u of the KP hierarchy with the conserved covariant generator $\varphi\varphi^*$ as follows [5, 6]:

$$u = -2\varphi\varphi^* = -2qr \tag{3.1}$$

where $q = \varphi$ and $r = \varphi^*$ are used for simplicity. The spectral problem (2.6) and its adjoint (2.13) then become

$$q_y = -(q_{xx} - 2q^2 r) \tag{3.2a}$$

$$r_y = (r_{xx} - 2qr^2) \tag{3.2b}$$

which is the generalized NS equation for the dynamical variables q and r. Inserting (3.1) into the linear equation governing the time evolution of the eigenfunction and its adjoint for the KP equation (1.1) (i.e. the equation in (2.7) and (2.14) with $n = 3$) and using (3.2) leads to the generalized MKdV equation

$$q_{t_3} = (q_{xxx} - 6qrq_x) \tag{3.3a}$$

$$r_{t_3} = (r_{xxx} - 6qrr_x). \tag{3.3b}$$

The resulting equations (3.2) and (3.3) are the first two non-trivial equations in the $1+1$ dimensional AKNS hierarchy. Conversely, if q and r solve both (3.2) and (3.3), then u given by (3.1) solves the KP equation (1.1). Exact solutions of the KP equation obtained in this way have been shown in [5].

In the following, we show that if q and r satisfy (3.2), the constraint (3.1) leads to the whole of the linear systems (2.7) and their adjoints (2.14) of the KP hierarchy to the whole of the AKNS hierarchy, and solutions of both (3.2) and the nth-order equation in the AKNS hierarchy give rise to solutions of the nth-order KP equation in (2.1).

For a scalar or an operator $F(u)$ depending on u, we define that

$$F(u)|_R = F(u)|_{u=-2qr} \tag{3.4}$$

is the scalar or operator depending on q and r, *where q and r satisfy the generalized NS equation (3.2).*

Lemma 1. For $a_n(u)$ and $b(u)$ given in (2.2)-(2.4), we have

$$a_n(u)|_R = -2\langle \bar{u}, J\bar{a}_n(\bar{u})\rangle \tag{3.5}$$

$$b(u)|_R = -2\langle \bar{u}, J\bar{b}(\bar{u})\rangle \tag{3.6}$$

where

$$\bar{u} = \begin{pmatrix} q \\ r \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.7}$$

$$\bar{a}_n(\bar{u}) = \begin{pmatrix} \bar{a}_n^{(1)} \\ \bar{a}_n^{(2)} \end{pmatrix} = \Phi^n \begin{pmatrix} -q \\ r \end{pmatrix} \tag{3.8}$$

$$\bar{b}(\bar{u}) = \begin{pmatrix} \bar{b}^{(1)} \\ \bar{b}^{(2)} \end{pmatrix} = 2y\bar{a}_3(\bar{u}) + x\bar{a}_2(\bar{u}) + 2\tau_1(\bar{u}) \tag{3.9}$$

$$\tau_1(\bar{u}) = \begin{pmatrix} -q_x + q\partial_x^{-1}qr \\ r_x - r\partial^{-1}qr \end{pmatrix} \tag{3.10}$$

and Φ is the recursion operator of the AKNS system

$$\Phi = \begin{pmatrix} -\partial_x + 2q\partial_x^{-1}r & 2q\partial_x^{-1}q \\ -2rq\partial_x^{-1}r & \partial_x - 2r\partial_x^{-1}q \end{pmatrix}. \tag{3.11}$$

In (3.5) and (3.6), $\langle , \rangle$ is defined as

$$\langle f, g\rangle = f^{(1)}g^{(1)} + f^{(2)}g^{(2)} \tag{3.12}$$

for any $f = (f^{(1)}, f^{(2)})^T$ and $g = (g^{(1)}, g^{(2)})^T$.

Proof. Notice that $\bar{b}(\bar{u})$ in (3.9) also generates $\bar{a}_n$ in the same way as $b(u)$ does in (2.3); it plays the role of mastersymmetry of the AKNS hierarchy (see e.g. [14]). By direct calculation, we can check (3.5) for $n=1$ and (3.6). If (3.5) holds for arbitrary n, then

$$\begin{aligned} a_n'[b]|_R &= \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon} a_n(u+\varepsilon b)|_R \\ &= \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon} a_n(-2(q+\varepsilon\bar{b}^{(1)})(r+\varepsilon\bar{b}^{(2)})) \\ &= \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon} -2\langle\bar{u}+\varepsilon\bar{b}, J\bar{a}_n(\bar{u}+\varepsilon\bar{b})\rangle \\ &= -2\langle\bar{u}, J\bar{a}_n'[\bar{b}]\rangle - 2\langle\bar{b}, J\bar{a}_n\rangle \end{aligned}$$

and similarly

$$b'[a_n]|_R = -2\langle\bar{u}, J\bar{b}'[\bar{a}_n]\rangle - 2\langle\bar{a}_n, J\bar{b}\rangle.$$

Thus for $n+1$, we find

$$\begin{aligned} a_{n+1}(u)|_R &= \frac{1}{n}(a_n'[b]-b'[a_n])|_R \\ &= -\frac{2}{n}\langle\bar{u}, J(\bar{a}_n'[\bar{b}]-\bar{b}'[\bar{a}_n])\rangle \\ &= -2\langle\bar{u}, J\bar{a}_{n+1}(\bar{u})\rangle. \end{aligned}$$ □

Lemma 2. For the Lax operators A_n in (2.9) and (2.10), B in (2.11) and spectral operator L in (2.6), let us define

$$U_n = \mathrm{diag}\,(A_n, -A_n^*) \qquad V = \mathrm{diag}\,(B, -B^*) \qquad T = \mathrm{diag}\,(L, L^*) \tag{3.13}$$

for simplicity, then

$$T|_R\tau_m = 0 \qquad \tau_m'[\bar{a}_n] = U_n|_R\tau_m \tag{3.14a, b}$$

where $\tau_1(\bar{u})$ is in (3.10), and $\tau_m(\bar{u})$, $m>1$ are defined recursively as follows:

$$\tau_{m+1}(\bar{u}) = \tau_m'[\bar{b}] - V|_R\tau_m. \tag{3.15}$$

Proof. We only need to prove the first component of (3.14), the other can be proved similarly. The first component of (3.14a) for $n=1$ can be checked easily since q and r satisfy (3.2), if we have $L|_R\tau_m^{(1)}=0$, then take the Gateaux derivative in the direction $\bar{b}(\bar{u})$ and note that

$$(L|_R)'[\bar{b}] = b(u)|_R = [B, L]|_R \tag{3.16}$$

we find

$$0 = L|_R(\tau_m^{(1)})'[\bar{b}] + [B, L]|_R\tau_m^{(1)} = L|_R((\tau_m^{(1)})'[\bar{b}] - B|_R\tau_m^{(1)}) = L|_R\tau_{m+1}^{(1)}.$$

Thus we have (3.14a) for all n.

To prove (3.14b), we need to calculate

$$
\begin{aligned}
A'_n[b]|_R &= \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon} A_n(u+\varepsilon b)|_R \\
&= \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon} A_n(-2(q+\varepsilon b^{(1)})(r+\varepsilon b^{(2)})) \\
&= \lim_{\varepsilon\to 0}\frac{\partial}{\partial\varepsilon} (A_n|_R)(\bar{u}+\varepsilon\bar{b}) \\
&= (A_n|_R)'[\bar{b}]
\end{aligned}
\tag{3.17}
$$

and similarly

$$
B'[a_n]|_R = (B|_R)'[\bar{b}] \tag{3.18}
$$

so for the first component in (3.14b), when $n=1$, it holds trivially. If $(\tau_m^{(1)})'|[\bar{a}_n] = A_n|_R\tau^{(1)}$ for the fixed n and arbitrary m, then for arbitrary m,

$$
\begin{aligned}
((\tau_m^{(1)})'[\bar{a}_n])'[\bar{b}] &= (A_n|_R)'[\bar{b}]\tau_m^{(1)} + A_n|_R(\tau_m^{(1)})'[\bar{b}] \\
&= (A'_n[b]+A_nB)|_R\tau_m^{(1)} + A_n|_R\tau_{m+1}^{(1)} \\
((\tau_m^{(1)})'[\bar{b}])'[\bar{a}_n] &= (B'[a_n]+BA_n)|_R\tau_m^{(1)} + (\tau_{m+1}^{(1)})'[\bar{a}_n]
\end{aligned}
$$

here we have used (3.15), (3.17) and (3.18). Since $\bar{b}$ generates $\bar{a}_n$ in the same way as b does in (2.3), and B generates A_n in (2.10), so the difference of the left-hand side of the above two equations gives rise to

$$
(\tau_m^{(1)})'(\bar{a}'_n[\bar{b}] - \bar{b}'[\bar{a}_n]) = n(\tau_m^{(1)})'[\bar{a}_{n+1}]
$$

and the difference of the right-hand side yields $nA_{n+1}|_R\tau_m^{(1)}$.

This implies that the first component in (3.14b) holds for $n+1$ and arbitrary m. □

Lemma 3. For U_n and V in (3.13) and $\bar{a}_n(\bar{u})$ in (3.8) we have

$$
U_n|_R\bar{u} = \bar{a}_n(\bar{u}) \tag{3.19}
$$

$$
V|_R\bar{u} = \bar{b}(\bar{u}) - \tau_1(\bar{u}). \tag{3.20}
$$

Proof. Equation (3.19) for $n=1,2$ and (3.20) can be checked directly. If (3.19) holds for n, then by using (3.17) and (3.18), we have

$$
\begin{aligned}
A_{n+1}|_Rq &= \frac{1}{n}(A'_n[b] - B'[a_n] + [A_n, B])|_Rq \\
&= \frac{1}{n}((A_n|_R)'[\bar{b}]q - (B|_R)'[\bar{a}_n]q + A_n|_R\bar{b}^{(1)} - B|_R\bar{a}_n^{(1)} - A_n|_R\tau_1) \\
&= \frac{1}{n}((A_n|_Rq)'[\bar{b}] - (B|_Rq)'[\bar{a}_n] - A_n|_R\tau_1) \\
&= \frac{1}{n}((\bar{a}_n^{(1)})'[\bar{b}] - (\bar{b}^{(1)})'[\bar{a}_n] + \tau'_1[\bar{a}_n] - A_n|_R\tau_1) \\
&= \bar{a}_{n+1}^{(1)}(\bar{u}).
\end{aligned}
$$

Thus the first component in (3.19) holds for all n. The second component of the equations (3.19) can also be proved in the same way. □

Theorem 1. When q and r satisfy (3.2), we have:

(i) *The linear systems* (2.7) *together with their adjoints* (2.14) *of the* KP *hierarchy* are constrained by (3.1) to the AKNS hierarchy

$$\bar{u}_{t_n} = \bar{a}_n(\bar{u}). \tag{3.21}$$

(ii) For any solution q and r of both (3.2) and the nth-order equation in (3.21), u given by (3.1) (i.e. $u = -2qr$) satisfies the nth-order equation in the KP hierarchy (2.1).

(iii) Corresponding to such potential $u = -2qr$ of the KP hierarchy $(\varphi, \varphi^*) = (q, r)$, *or* $= \tau_m$ *in* (3.15) *are the eigenfunctions and the adjoint eigenfunctions, namely two* components of $\bar{u}$ and $\tau_m(\bar{u})$ solve (2.7) and (2.14) respectively, with respect to the potential $u = 2qr$.

The first and third facts in the theorem are the results of lemma 3 and lemma 2. For the second fact, we have

$$(u_{t_n} - a_n(u))|_R = -2\langle \bar{u}, J(\bar{u}_{t_n} - \bar{a}_n(\bar{u}))\rangle.$$

The right-hand side is zero implies $u = -2qr$ solves (2.1).

In [5], soliton-like solutions and the solutions periodically along the x-axis of the KP equation (1.1) have been derived from the solitons and periodic solution of the $1+1$ dimensional system consisting of (3.2) and (3.3).

There are some other consequences. Firstly, lemma 1 gives the correspondence between symmetries of KP and AKNS hierarchies. The conserved covariants

$$\gamma_n(u) = \partial_x^{-1} a_n(u) \tag{3.22}$$

for the KP hierarchy and

$$\bar{\gamma}_n(\bar{u}) = \theta \bar{a}_n(\bar{u}) \qquad \theta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.23}$$

for the AKNS hierarchy are related by

$$\frac{\partial}{\partial x}(\gamma_n(u)|_R) = -2\langle \bar{u}, \sigma_3 \bar{\gamma}_n(\bar{u})\rangle \tag{3.24}$$

where σ_3 is the Pauli matrix. This means the conserved covariants (3.22) of the KP hierarchy are restricted to the conserved densities of the AKNS systems.

Secondly, the stationary equations

$$a_n(u) = 0 \tag{3.25}$$

of the KP *hierarchy* are constrained by (3.1) to the system consisting of (3.2) and the stationary equation of (3.21). The latter is a finite-dimensional integrable system (see e.g. [15]). For example, when $n = 3$ the stationary equation $\bar{a}_3(\bar{u}) = 0$ (i.e. the stationary generalized mKdV of (3.3)) can be written as

$$q_{ix} = \frac{\delta H_3}{\delta p_i} \qquad p_{ix} = -\frac{\delta H_3}{\delta q_i} \qquad i = 1, 2, 3 \tag{3.26}$$

with Hamiltonian

$$H_3 = (p_2 p_3 + q_2 p_1 - 3 q_1 q_2 q_3^2) \tag{3.27}$$

and the dynamical variables

$$\begin{aligned} &q_1 = q \qquad q_2 = q_x \qquad q_3 = r \\ &p_1 = -r_{xx} + 3qr^2 \qquad p_2 = r_x \qquad p_3 = q_{xx}. \end{aligned} \tag{3.28}$$

According to these variables, (3.2) is also the equation of motion of a finite-dimensional Hamiltonian, i.e.

$$q_{iy}=\frac{\delta H_2}{\delta p_i} \qquad p_{iy}=-\frac{\delta H_2}{\delta q_i} \qquad i=1,2,3 \tag{3.29}$$

with

$$H_2=(q_1q_3^2p_3-2q_1q_2q_3p_2-2q_1^3q_3^3+2q_1^2p_1q_3+q_2^2q_3^2+q_1^2p_2^2-p_1p_3). \tag{3.30}$$

Thus the stationary KP equation

$$a_3(u)=\tfrac{3}{4}\partial_x^{-1}u_{yy}+\tfrac{1}{4}u_{xxx}+\tfrac{3}{2}uu_x=0 \tag{3.31}$$

which is also called the Boussinesq equation in 1+1 dimensions, and the associated linear systems are constrained by (3.1) to the finite-dimensional integrable Hamiltonian systems (3.26) and (3.29).

4. Constraint of KP hierarchy to Burgers hierarchy

In this section we show that the KP hierarchy can also be constrained to the linearizable Burgers hierarchy, and so a submanifold solution of the KP hierarchy can be obtained by solving the linear equations. Let us identify the potential u with the eigenfunctions φ as follows:

$$u=2\varphi_x=2q_x \tag{4.1}$$

the spectral problem (2.6) is then reduced to

$$q_y+q_{xx}+2qq_x=0 \tag{4.2}$$

which is the Burgers equation. Substituting (4.1) into (2.7) for $n=3$ and using (4.2), we obtain the following third-order equation in the Burgers hierarchy:

$$q_{t_3}=q_{xxx}+3qq_{xx}+3q_x^2+3q^2q_x. \tag{4.3}$$

Conversely, if q solves both (4.2) and (4.3), then $u=2q_x$ can be checked directly to solve the KP equation (1.1). Equations (4.2) and (4.3) are linearizable by the Cole–Hopf transformation

$$q=\frac{\tau_x}{\tau} \tag{4.4}$$

i.e. they are transformed by (4.4) to

$$\tau_y=-\tau_{xx} \qquad \tau_{t_3}=\tau_{xxx} \tag{4.5}$$

respectively. Thus the solution of linear system (4.5) gives rise to the solution

$$u=2q_x=2\frac{\partial^2}{\partial x^2}\ln\tau \tag{4.6}$$

of the KP equation (1.1), which is in the same form as that in the 'τ-function' theory. The exact solutions obtained in terms of such constraint have been shown in [7]. In the following we investigate the same constraint for the whole of the KP hierarchy.

Denoting

$$F(u)|_{R_1}=F(u)|_{u=2q_x} \tag{4.7}$$

for a scalar or an operator $F(u)$, *where q satisfies the Burgers equation (4.2).*

Lemma 4. For $a_n(u)$, $b(u)$ in (2.2)-(2.4) and the Lax operators $A_n(u)$, $B(u)$ in (2.9)-(2.11), we have

$$a_n|_{R_1} = 2\partial_x(\tilde{a}_n(q)) \tag{4.8}$$

$$b|_{R_1} = 2\partial_x(\tilde{b}(q)) \tag{4.9}$$

and

$$A_n|_{R_1} q = \tilde{a}_n(q) \tag{4.10}$$

$$B|_{R_1} q = \tilde{b}(q) \tag{4.11}$$

where

$$\tilde{a}_n(q) = \Psi^{n-1} q_x \tag{4.12}$$

$$\tilde{b}(q) = 2y\tilde{a}_3(q) + x\tilde{a}_2(q) - 2q_x - q^2 + q_x \tag{4.13}$$

and

$$\Psi = -(\partial_x + q + q_x\partial_x^{-1}) \tag{4.14}$$

is the recursion operator of the following Burgers hierarchy:

$$q_{t_n} = \tilde{a}_n(q). \tag{4.15}$$

Proof. Note that the term $x\tilde{a}_2(q) - 2q_x - q^2$ in (4.13) is the mastersymmetry of the Burgers hierarchy (4.15); it generates $\tilde{a}_n$ in the same way as $b(u)$ does in (2.3). So $\tilde{b}(q)$ in (4.13) plays the same role, namely $\tilde{b}(q)$ also generates $\tilde{a}_n$ in the same way as $b(u)$ does in (2.3) (see e.g. [16, 17]). Therefore one can firstly check (4.8) and (4.10) for $n = 1$, and (4.9) and (4.11), by direct calculation. Then by induction (4.8) and (4.11) for arbitrary n can be proved. □

So we conclude that when q solves the Burgers equation (4.2), then:

(i) The linear systems (2.7) of the KP hierarchy are constrained by (4.1) to the Burgers hierarchy (4.15).

(ii) For any solution q of both (4.2) and (4.15), $u = 2q_x$ solves the equation (2.1) of the KP hierarchy. Since (4.15) can be transformed by (4.4) to the linear equations for τ, so the solution of the KP hierarchy can also be obtained in the form (4.6) with τ satisfying

$$\tau_y = -(\partial_x^2\tau) \qquad \tau_{t_n} = (-1)^{n-1}(\partial_x^n\tau). \tag{4.16}$$

Remarks. It is worth noting that firstly in the Sato theory, the τ-function of the KP hierarchy can be expressed as the determinant with the elements satisfying the linear equations (see e.g. [18]). Our second result in the conclusion probably coincides with this fact in the Sato theory. Secondly, by using the dressing method, a class of solutions of the KP equation can be derived in terms of the solutions of the linear systems (4.5) (see e.g. [19]). The soliton-like solutions obtained from the constraint (4.1) is a special case of the known ones derived by the dressing method as well as the Hirota bilinear method [7].

5. Constraints for other soliton equations in 2+1 dimensions

Let us consider the following modified KP equation:

$$v_t = \tfrac{1}{4}(v_{xxx} - 6v^2 v_x - 6v_x \partial_x^{-1} v_y + 3\partial_x^{-1} v_{yy}). \tag{5.1}$$

It is the compatibility condition of [20]

$$\varphi_y + \varphi_{xx} + 2v\varphi_x = 0 \tag{5.2}$$

$$\varphi_t = \varphi_{xxx} + 3v\varphi_{xx} + \tfrac{3}{2}(v_x + v^2 - \partial_x^{-1} v_y)\varphi_x. \tag{5.3}$$

The conserved covariant generator in this case is $\varphi\psi$ where $\psi = (\varphi_x)^*$ satisfying

$$\psi_y - \psi_{xx} + 2v\psi_x = 0 \tag{5.4}$$

$$\psi_t = \psi_{xxx} - 3v\psi_{xx} - \tfrac{3}{2}(v_x - v^2 + \partial_x^{-1} v_y)\psi_x. \tag{5.5}$$

If we identify u with the conserved covariant generator $\varphi\psi$ as follows:

$$v = \varphi\psi = qr \tag{5.6}$$

where $q = \varphi$, $r = \psi$ are used for simplicity, then substituting (5.6) into (5.2) and (5.4) yields

$$q_y = -q_{xx} - 2qrq_x \tag{5.7a}$$

$$r_y = r_{xx} - 2qrr_x \tag{5.7b}$$

while (5.3) and (5.5) are constrained to

$$q_t = q_{xxx} + 3qrq_{xx} + 3(q^2r^2 + q_x r)q_x \tag{5.8a}$$

$$r_t = r_{xxx} - 3qrr_{xx} + 3(q^2r^2 - qr_x)r_x. \tag{5.8b}$$

Equation (5.7) is the generalized NS equation with derivative coupling given by Chen *et al* [21] and (5.8) is the higher-order equation of (5.7). Both of them are Hamiltonian with

$$H_2 = \int (qr_{xx} - q^2(r^2)_x)\,\mathrm{d}x \tag{5.9}$$

$$H_3 = \int (qr_{xxx} + q^3(r^3)_x - 2q^2(r^2)_{xx})\,\mathrm{d}x. \tag{5.10}$$

Their integrability has been shown in [21], and the gauge equivalence to the generalized derivative NS equation can be obtained [22].

One can also check that when q and r solve (5.7) and (5.8), $v = qr$ solves the mKP (5.1).

By the Miura transformation of [20]

$$u = -(\partial_x^{-1} v_y + v_x + v^2) \tag{5.11}$$

the solution of mKP in the form of (5.6) is transformed to the solution of KP (1.1) $u = -2qr_x$, namely once q and r solve both (5.7) and (5.8), $u = -2qr_x$ solves the KP equation.

Next, if we insert

$$v = \varphi \equiv q \tag{5.12}$$

into (5.2) and (5.3) respectively, we again obtain the Burgers and the third-order Burgers equation (4.2) and (4.3), and conversely the solution of both (4.2) and (4.3) also solves the mKP (5.1).

This solution is transformed by (5.11) to the trivial solution of the KP.

Let us now consider a more complicated example given in [18],

$$u_t = \tfrac{5}{9}(\partial_x^{-1} u_{yy} - u_x \partial_x^{-1} u_y - \tfrac{1}{5} u_{xxxxx} - u_x u_{xx} - u u_{xxx} - u^2 u_x - u_{xxy} - u u_y). \tag{5.13}$$

This is the 2+1 dimensional analogue of the CDGKS equation and is the compatibility condition of the following linear systems [20]:

$$\varphi_y + \varphi_{xxx} + u\varphi_x = 0 \tag{5.14}$$

$$\varphi_t = \varphi_{xxxxx} + \tfrac{5}{3}(u\varphi_{xxx} + u_x\varphi_{xx}) + \tfrac{5}{9}(2u_{xx} + u^2 - \partial_x^{-1} u_y)\varphi_x. \tag{5.15}$$

Firstly, if we identify u with φ^2 as follows:

$$u = -6\varphi^2 = -6q^2 \tag{5.16}$$

(5.14) and (5.15) become

$$q_y + q_{xxx} - 6q^2 q_x = 0 \tag{5.17}$$

$$q_t = q_{xxxxx} - 10q^2 q_{xxx} - 40 q q_x q_{xx} + 30 q^4 q_x - 10 q_x^3 \tag{5.18}$$

which are the mKdV and the higher-order equation next to mKdV in the same hierarchy. Thus a submanifold solution of (5.13) can be obtained by solving both (5.17) and (5.18).

Secondly, inserting

$$u = 6\varphi \equiv 6p \tag{5.19}$$

into (5.14) and (5.15), we have

$$p_y + p_{xxx} + 6pp_x = 0 \tag{5.20}$$

$$p_t = p_{xxxxx} + 10pp_{xxx} + 20p_x p_{xx} + 30p^2 p_x \tag{5.21}$$

which is the KdV and the fifth-order KdV equations. The solution p of both (5.20) and (5.21) also gives rise to solution $u = 6p$ of (5.13).

The equations (5.17) and (5.18) can be transformed by the Miura transformation

$$p = -q_x - q^2 \tag{5.22}$$

to (5.20) and (5.21) respectively. If a solution $u = -6q^2$ of the 2+1 dimensional CDGKS equation (5.13) is obtained by the first constraint, then by using the Miura transformation (5.22), we find

$$\bar{u} = 6p = -6q_x - 6q^2 \tag{5.23}$$

is the solution of (5.13) which can be obtained by the second constraint. Equation (5.23) implies that $\bar{u}$ can be expressed in terms of u, i.e.

$$\bar{u} = \pm \frac{\sqrt{6}}{2} \frac{u_x}{\sqrt{-u}} + u. \tag{5.24}$$

This is the Backlund transformation of the equation (5.13), which transforms solutions obtained by the first constraint to the solutions obtained by the second constraint for the same equation (5.13). It is not yet clear whether (5.24) gives the general Backlund transformation of (5.13).

Remarks. All the constraints for mKP, or 2+1 dimensional CDGKS equations, can be generalized to their higher-order equations in the same way as we have shown for the KP hierarchy, since the mastersymmetries and their correspondent Lax operators have been given in [23].

6. Concluding remarks

We have shown that some 2+1 dimensional integrable hierarchies of the soliton equations can be constrained to the well known 1+1 dimensional integrable systems. One of the important consequences is that varieties of solutions of the equations in 2+1 can be obtained by solving the resulting integrable systems in 1+1 dimensions. The constraint for each equation discussed in this paper is not unique. It is, therefore, natural to ask whether there exists other constraints for a 2+1 dimensional integrable system playing the same role as those given in the present paper, or resulting some new integrable systems in 1+1 dimensions. This question probably relies on the problem of the existence of a role for making a constraint in the sense that the integrability of the given 2+1 dimensional soliton equation is inherited by the resulting systems in 1+1 dimensions.

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